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Natural pseudodistance for shape comparison and quotients of metric spaces

Abstract. *A general topological setting to describe the notion of natural pseudodistance is introduced. Conditions to get a distance are given.*

Key Words: *Size functions, Natural pseudodistance, Pseudometric spaces.*

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1. Introduction

Size functions are a mathematical tool for shape comparison in computer vision and pattern recognition.

Size functions study the shape of an object by a pair (X, φ), where X is a topological space and $\varphi : X \to \mathbb{R}$ is a continuous real function. In this situation, X represents the set of observations made by the observer, while φ describes how each observation is interpreted by the observer. (X, φ) is called a *size pair*, while the function φ is called a *measuring function*.

In this setting, the notion of *natural pseudodistance* has been introduced in [4] in order to measure the dissimilarity between the perceptions expressed by the pairs $(X, \varphi), (Y, \psi)$, when X and Y are compact manifolds:

if (X, φ) and (Y, ψ) are size pairs on compact manifolds, and $\mathcal{H}$ is the set of homeomorphisms between X and Y, the *natural pseudodistance* between

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φ and ψ is:

$$\delta(\varphi,\psi)=\inf\{\max_{P\in X}|\varphi(P)-\psi(h(P))|, h\in\mathcal{H}\}.$$

If this infimum is a minimum, the homeomorphism h which realizes this minimum is called an *optimal homeomorphism*.

The natural pseudodistance is worthy only if the two manifolds are homeomorphic, otherwise the pseudodistance between two maps is always ∞.

The notion of pseudodistance can be rephrased in a different way. First of all we can assume without loss of generalty that the two manifolds X and Y are the same manifold: with a fixed homeomorphism $\bar{h}$ between X and Y, any measuring function ψ on Y will correspond to the map $\psi\bar{h}$ on X.

Moreover if $X=Y$ and $\varphi'=\varphi h, \psi'=\psi k$, $h,k\in\mathcal{H}$ we have $\delta(\varphi',\psi')=\delta(\varphi,\psi)$.

Therefore the natural pseudodistance can be defined on $C(X,\mathbb{R})/\sim$ the quotient of the metric space $(C(X,\mathbb{R}),d)$, (d is the metric of uniform convergence) under the action of $\mathcal{H}$, setting $\delta_{\mathcal{H}}(\varphi,\psi)=d([\varphi]_\sim,[\psi]_\sim)$.

In addition instead of taking the whole group $\mathcal{H}$ of homeomorphisms of X, we can consider a subgroup $\mathcal{K}$ of $\mathcal{H}$ and $\delta_{\mathcal{K}}$, the pseudodistance w.r.t. $\mathcal{K}$, defined as δ using only homeomorphisms in $\mathcal{K}$. We will prove that $\delta_{\mathcal{K}}$ is always the quotient pseudo-metric induced by the action of $\mathcal{K}$ on $C(X,\mathbb{R})$.

$\delta_{\mathcal{K}}$ is not in general a distance, that is there could be measuring functions $\varphi,\psi\in C(X,\mathbb{R})$ s.t. $\delta_{\mathcal{K}}(\varphi,\psi)=0$, but with no $k\in\mathcal{K}$ for which $\psi=\varphi k$.

In this paper conditions under which the natural pseudodistance becomes a distance are given.

2. Known results

Prior works have assessed the following properties, under the assumption that the measuring functions are C^1:

1 If d denotes the natural pseudodistance between φ and ψ, then a positive integer k exists for which one of the following statements holds:
- k is odd and kd equals the distance between a critical value of φ and a critical value of ψ.
- k is even and kd equals either the distance between two critical values of φ or the distance between two critical values of ψ [3, 4].

2 If an optimal homeomorphism $\widetilde{h}\in\mathcal{H}$ between φ and ψ exists, then the natural pseudodistance between φ and ψ equals the distance between a critical value of φ and a critical value of ψ [4].

3 Let $(X, \varphi), (Y, \psi)$ be two size pairs, with X, Y closed C^2 curves, and $\varphi : X \to \mathbb{R}$ and $\psi : Y \to \mathbb{R}$ Morse measuring functions. If $\delta(\varphi, \psi) = 0$, then there exists an optimal C^2 diffeomeomorphism $f : X \to Y$ [2].

3. Quotients of pseudometric spaces

PMet is the category of pseudometric spaces and non-expansive maps.

A map $f : (Z, d) \to (Z', d')$ between pseudometric spaces is non-expansive if for all $z_1, z_2 \in Z$

$$d'(f(z_1), f(z_2)) \leq d(z_1, z_2).$$

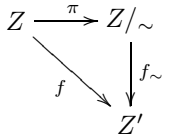
We need non-expansive maps because this category is the best behaved for our purposes. Indeed, in **PMet** we can form quotients.

If (Z, d) is a pseudometric space and $\sim$ is an equivalence relation on Z, we can endow the quotient set $Z/_\sim$ with the following pseudodistance. Given two equivalence classes $[z]$ and $[y]$, the *quotient pseudometric* is defined by

$$d_\sim([z], [y]) = \inf \left\{ \sum_{i=1}^{n} d(z_i, y_i) \right\}$$

where the infimum is taken over all finite sequences $(z_1, z_2, \dots z_n)$ and $(y_1, y_2, \dots y_n)$ with $[z_1] = [z], [y_{i-1}] = [z_i], [y_n] = [y],\ (i = 2, \dots, n)$ (see [1, page 137]).

The quotient pseudodistance $d_\sim$ is characterized by the following universal property. If $f : (Z, d) \to (Z', d')$ is a non-expansive map between pseudometric spaces such that $f(z) = f(y)$ whenever $z \sim y$, then we have that the induced quotient map $f_\sim : (Z/_\sim, d_\sim) \to (Z', d')$ is non-expansive, that is the following diagram commutes in **PMet**:



In general the quotient topology induced by $\sim$ is different from the topology induced by the quotient pseudodistance $d_\sim$.

In [5] a condition is given under which these two topologies coincide:

THEOREM 3.1. *If (Z, d) a pseudometric space endowed with an equivalence relation where the equivalence classes are the orbits of the action of a group of isometries on (Z, d), then*

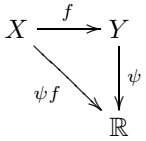
- $d_\sim([z],[y]) = \inf\{d(z',y') : z' \sim z; y' \sim y\}$;
- *the topology induced by the quotient pseudometric coincides with the quotient topology.*

As a consequence, in this case, the quotient pseudometric is a metric if and only if the orbits are closed sets of Z. (see Proposition 4.1 in the next section)

4. Natural pseudodistance as quotient distance

Coming back to our situation, we consider the function space $C(X,\mathbb{R})$ of real continuous maps, endowed with the compact open topology, with domain X in the category $\mathbf{M}$ of compact manifolds. Moreover $C(X,\mathbb{R})$ is a metric space with the uniform convergence metric d. Now we can consider the controvariant functor $F : \mathbf{M} \to \mathbf{PMet}$ so defined:

- $F(X) = C(X,\mathbb{R})$;
- if $f : X \to Y$ is a continuous map between compact manifolds, $F(f) : C(Y,\mathbb{R}) \to C(X,\mathbb{R})$ where $F(f)(\psi) = \psi f$



Therefore any homeomorphism between X and Y (any self-homeomorphism of X), induces an isometry between $C(Y,\mathbb{R})$ and $C(X,\mathbb{R})$ (on $C(X,\mathbb{R})$).

The orbits induced by the action of $\mathcal{K} \subseteq \mathcal{H} = Homeo(X)$ on $C(X,\mathbb{R})$ coincide with the equivalence classes:

$$[\varphi]_{\mathcal{K}} = \{\eta : \eta = \varphi h, h \in \mathcal{K}\}.$$

and

$$\delta_{\mathcal{K}}([\varphi]_{\mathcal{K}},[\psi]_{\mathcal{K}}) = \inf\{d(\varphi',\psi'), \varphi' \in [\varphi]_{\mathcal{K}}, \psi' \in [\psi]_{\mathcal{K}}\}$$

Now applying Theorem 3.1 we get the following results:

PROPOSITION 4.1. *The following statements hold:*

1. $\delta_{\mathcal{K}}$ *is the quotient pseudometric induced by the action of* $\mathcal{K}$ *on* $C(X,\mathbb{R})$, *this means* $\pi : (C(X,\mathbb{R}),d) \to (C(X,\mathbb{R})/\sim, \delta_{\mathcal{K}})$, $\pi(\varphi) = \pi(\psi)$ *if and only if in* $\mathcal{K}$ *there is a homeomorphism* $k : X \to X$, *such that* $\varphi = \psi k$ *is a quotient in* **PMet**;
2. $\pi : (C(X,\mathbb{R}),d) \to (C(X,\mathbb{R})/\sim, \delta_{\mathcal{K}})$ *is a topological quotient;*

3. $(C(X,\mathbb{R})/\sim, \delta_{\mathcal{K}})$ *is a metric space if and only if any orbit induced by* $\mathcal{K}$ *in* $C(X,\mathbb{R})$ *is closed.*
4. *If we take a compact subgroup* $\mathcal{K}$ *of* $\mathcal{H}$ *endowed with the compact open topology, and consider the action of* $\mathcal{K}$ *on* $C(X,\mathbb{R})$*, the space of the orbits is a metric space.*

If we replace $C(X,\mathbb{R})$ with its subspace $Morse(X,\mathbb{R})$ of Morse map from X to $\mathbb{R}$, the result of Proposition 4.1 still holds. So the theorems proved in [2] can be rewritten as follows:

PROPOSITION 4.2. *Any orbit in* $Morse(S^1,\mathbb{R})$ *induced by the group* $\mathcal{D}(H)$ *of diffeomorphisms of the circle* S^1 *is closed* [2].

REMARK 4.3. As an application of condition 4 in Proposition 4.1 we can consider the manifold X with the following properties:

- symmetric w.r.t. an hyperplane;
- manifold of revolution;
- invariant w.r.t. a rotation of $\dfrac{2\pi}{n}$

In each of the previous case or in each combination of them we can consider $\mathcal{K} = \mathbf{Z}_2; \mathbf{S}^1; \mathbf{Z}_n$ or the corresponding product of these compact groups. Then the orbits $C(X,\mathbb{R})$ induced by $\mathcal{K}$ are closed so that $\delta_{\mathcal{K}}$ is a metric on $C(X,\mathbb{R})/\sim$.

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